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On commutator ideals in free Lie algebras

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Abstract

Let L be a free Lie algebra over a field k, I and J non-trivial proper ideals of L. If $I + J \neq L$ then the Schur multiplier, $H_2(L/[I, J], k)$, of $L/[I, J]$ is not finite dimensional, and so in particular, $L/[I, J]$ is not finitely presented.

1. Introduction

If R is a free associative algebra, over a field, and I is a two sided ideal of R, then Lewin proved [10] that I^2 is not finitely generated (as a 2-sided ideal!) when the algebra R/I is infinite dimensional. In other words, R/I^2 is not finitely presented in this case. On the other hand, it is easy to see that when R is finitely generated and R/I is finite dimensional, so is R/I^2, and hence I^2 is finitely generated. In fact, if R/I is an infinite-dimensional domain, then for $n \geqslant 2$, $\mathrm{Tor}_2^{R/I^n}(k, k)$ is infinite dimensional [2].

Similar behavior is seen in groups. If F is a finitely generated free group, and R is a normal subgroup then R' is *normally* finitely generated if, and only if, F/R is finite. In fact Baumslag, Strebel and Thomson proved [6] a stronger fact. Denoting the mth member of the lower central series by γ_m, they proved that for $m > 1$ the Schur multiplier of $F/\gamma_m R$, $H_2(F/\gamma_m R, \mathbb{Z})$, is not finitely generated (as an abelian group) if F/R is not finite.

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In Lie algebras a similar result can be shown. If L is a finitely generated free Lie algebra over a field k, and I is a non-zero proper ideal of L (note that it is not required that L/I be infinite dimensional), then for $n \geqslant 2$, $H_2(L/I^n, k)$ is infinite dimensional, where $I^1 = I$ and $I^m = [I^{m-1}, I]$ for $m > 1$ [4].

In [3] a question was considered regarding the Schur multiplier of $F/[R, S]$ where R, S are subgroups of the free group F (the structure of $F/[R, S]$ was studied in [8], though not the question whether it is finitely presented). It was shown that for most cases, if F/RS is infinite then the Schur multiplier of $F/[R, S]$ is not finitely generated (the authors believe this to be true in all cases). It was also shown that if RS is of finite index, and both R, S are normally finitely generated, then $[R, S]$ is normally finitely generated. Finally, an example was given where R is not normally finitely generated, yet $[R, F]$ is.

In this paper we prove a result of similar nature for Lie algebras.

Theorem 1.1. *Let L be a free Lie algebra with basis X, over a field k, and let I, J be* any *non-zero proper ideals of L. If $I + J = L$ and I, J are finitely generated as ideals, then $[I, J]$ is finitely generated as an ideal. If $I + J \neq L$ then $[I, J]$ is not finitely generated as an ideal. In fact, the Schur multiplier of $L/[I, J]$, $H_2(L/[I, J], k)$, is not finite dimensional.*

In Section 2 we define some notations and the Magnus embedding. In Section 3 we build a mapping from the Schur multiplier into a tensor product of copies of $U(L/(I + J))$. In Section 4 we build a specific isomorphism of Hopf modules, keeping in mind that the enveloping algebra of a Lie algebra is a Hopf algebra. In Section 5 we explore the Schur multiplier of certain Lie algebras over $\mathbb{Z}$ that will be needed for fields of characteristic 2. In Section 6 we employ the mapping and show that the image of the Schur multiplier is not finite dimensional, thus proving the theorem, except for certain cases with characteristic 2. In Section 7 we finish the proof by dealing with these special cases. In Section 8 we show the case where $I + J = L$ and give an example of an ideal I that is not finitely generated as an ideal, yet $[I, L]$ is finitely generated.

2. Preliminaries and notations

Let $\mathcal{G}$ be a Lie algebra. We will denote the Lie multiplication of two elements $a, b \in \mathcal{G}$ by $[a, b]$. As we will also be considering the enveloping algebra of $\mathcal{G}$, the multiplication in $U(\mathcal{G})$ will be denoted simply as ab, while the action of an element $l \in U(\mathcal{G})$ on an element $a \in \mathcal{G}$ will be denoted by $a \cdot l$. Note that the action is the adjoint action, so that if $l \in \mathcal{G}$ then $a \cdot l = [a, l]$.

Let $\mathcal{G}$ be a Lie algebra over a field k, $U(\mathcal{G})$ its enveloping algebra, $\delta U(\mathcal{G})$ the augmentation ideal of $U(\mathcal{G})$. Suppose $0 \to I \to L \to \mathcal{G} \to 0$ is a free presentation of $\mathcal{G}$, where L is the free Lie algebra over k with basis X. The enveloping algebra, $U(L)$, is therefore a free associative algebra, with basis X, and $\delta U(L)$ is a free $U(L)$ module, with a basis in one-to-one correspondence with X. Note that over a field, if $\mathcal{G} \neq 0$, $U(\mathcal{G})$ is infinite dimensional, and is without zero divisors.

In addition, if $\mathcal{G}$ is a Lie algebra over a field and $U(\mathcal{G})$ is its enveloping algebra, let $U_n(\mathcal{G})$ be the subspace of $U(\mathcal{G})$ spanned by all the products of at most n factors from $\mathcal{G}$.

This gives a well-known ascending filtration of $U(\mathcal{G})$, and we can define the *degree* of an element l to be the *least* integer n such that $l \in U_n(\mathcal{G})$. This function has the properties:

(1) $\deg(a+b) \leqslant \max\{\deg(a), \deg(b)\}$,
(2) if $\deg(a) < \deg(b)$ then $\deg(a+b) = \deg(b)$,
(3) $\deg(ab) = \deg(a) + \deg(b)$.

In particular, if $x \in \mathcal{G}$ is non-zero then the degree of x is 1, so if $x_1, x_2, \ldots, x_n \in \mathcal{G}$ are all non-zero then $\deg(x_1 x_2 \cdots x_n) = n$.

If $a \in U(\mathcal{G})$, and $\deg(a) = m$ then we will write $\bar{a}$ for the image of a in $U_m(\mathcal{G})/U_{m-1}(\mathcal{G})$, and will call it the leading term of a.

If K is an ideal of L then K/K' carries the structure of a $U(L)$ module, via the adjoint action, and K acts trivially. All modules will be right modules. Therefore K/K' is a $U(L/K)$ module in a natural way. There is a well-known embedding of $U(L/K)$ modules, the Magnus embedding, described below, of K/K' into $\delta U(L) \otimes_{U(L)} U(L/K)$. This embedding will be denoted by $\phi : K/K' \to \delta U(L) \otimes_{U(L)} U(L/K)$. This is of course an analogue of the Magnus embedding in groups. The action of L on $\delta U(L) \otimes_{U(L)} U(L/K)$ is by right multiplication on the right-hand term.

The embedding $\phi : K \to \delta U(L) \otimes_{U(L)} U(L/K)$ can be defined by $\phi(x) = x \otimes 1$. As is well known (see, e.g., [7]), ϕ is a $U(L)$ module mapping, and its kernel is exactly K'.

Throughout the remainder of this paper I, J will be proper non-zero ideals of L, and $K = I + J$.

3. An image of $H_2(L/[I, J], k)$

Consider $H_2(L/[I, J], k)$. It is known (e.g., [12, p. 233]) that the analogue of the Hopf formula for groups holds for Lie algebras. Thus

$$H_2\big(L/[I, J], k\big) = [I, J]/\big[[I, J], L\big].$$

We know from the Širšov–Witt theorem (see, e.g., [11, p. 44]) that K is a free Lie algebra. Hence K^n/K^{n+1} is, in a natural way, identifiable with the nth homogeneous component of the free Lie algebra with basis that is a basis of K/K' as a vector space. Since a free Lie algebra can be embedded in the tensor algebra over a vector space with basis in one-to-one correspondence with the Lie algebra's basis, the nth homogeneous component can be embedded into the n-fold tensor product, thus K^n/K^{n+1} can be embedded in $\bigotimes^n K/K'$, where the tensor is over k. Any unadorned tensor product below is to be taken to be over k. We need this embedding to be a $U(L/K)$ module homomorphism, and it is easy to see that this is indeed the case when $U(L/K)$ acts on K^n/K^{n+1} via the adjoint action, and on $\bigotimes^n K/K'$ diagonally. Note that unlike groups, the diagonal action is not a simultaneous application to all the components, but rather $(a \otimes b) \cdot x = (a \cdot x \otimes b) + (a \otimes b \cdot x)$ if x is a Lie element. The module $\bigotimes^n K/K'$ can again can be embedded, through the Magnus embedding, into

$$\bigotimes^{n}\big(\delta U(L)\otimes_{U(L)}U(L/K)\big).$$

It can be easily checked that if $\alpha\in K$ and $\beta\in K^n$ then the image of $[\alpha,\beta]$ in $\bigotimes^{n+1}(\delta U(L)\otimes_{U(L)}U(L/K))$ will be $a\otimes b-b\otimes a$ where a is the image of α in $\delta U(L)\otimes_{U(L)}U(L/K)$ and b is the image of β in $\bigotimes^{n}(\delta U(L)\otimes_{U(L)}U(L/K))$.

Since K is itself a free Lie algebra, and free Lie algebras are residually nilpotent, there are numbers m,n such that $I\subseteq K^m$, $I\nsubseteq K^{m+1}$, $J\subseteq K^n$, $J\nsubseteq K^{n+1}$. Since $K\neq K'$ it is not possible that both $m>1$ and $n>1$ so we can assume without loss of generality that $m=1$. Thus, $[I,J]\subseteq K^{n+1}$, so we can build a natural mapping

$$[I,J]/\big[[I,J],L\big]\to K^{n+1}/\big[K^{n+1},L\big]=K^{n+1}/K^{n+2}\otimes_{U(L/K)}k.$$

Combining this with the Magnus embedding we get a mapping

$$H_2\big(L/[I,J],k\big)\to\bigotimes^{n+1}\big(\delta U(L)\otimes_{U(L)}U(L/K)\big)\otimes_{U(L/K)}k.$$

Since $\delta U(L)$ is a free $U(L)$ module, with a basis in one-to-one correspondence with X, the basis of L as a Lie algebra, we can define for each $x\in X$ a projection which assigns to an element its xth coordinate, denoted $p_x:\delta U(L)\otimes_{U(L)}U(L/K)\to U(L/K)$. We therefore have for each $(n+1)$-tuple $(x_1,x_2,\dots,x_{n+1})\in X^{n+1}$ a mapping $\phi_{x_1,\dots,x_{n+1}}:=(p_{x_1}\otimes\cdots\otimes p_{x_{n+1}}\otimes 1)\circ\phi$

$$\phi_{x_1,x_2,\dots,x_{n+1}}:H_2\big(L/[I,J],k\big)\to\bigotimes^{n+1}U(L/K)\otimes_{U(L/K)}k.$$

Since $K/K'\to\delta U(L)\otimes U(L/K)$ is an embedding, and $I\nsubseteq K'$, there exist elements $\alpha\in I$ and $x\in X$ such that under the Magnus embedding and the projection by x the image $a=p_x(\alpha)$ is non-zero. Since $J\nsubseteq K^{n+1}$ there also exist elements $\beta\in J$, $x_1,\dots,x_n\in X$ such that $b=(p_{x_1}\otimes\cdots\otimes p_{x_n})(\beta)$ is non-zero. These elements will be put to use below.

4. Isomorphism of Hopf modules

As seen in the last section, the image of the Schur multiplier lies in

$$M=\Bigg(\bigotimes^{n+1}U(L/K)\Bigg)\otimes_{U(L)}k.$$

On the other hand, it is well known that the enveloping algebra $U(L/K)$ is a Hopf algebra, and the action with which M is endowed is consistent with the standard Hopf structure on $U(L/K)$, which is the diagonal action. We shall use the following notation for the structure of Hopf algebras. Let H be a Hopf algebra. The diagonal mapping of H will be denoted by Δ, and the n-fold application of Δ by Δ_n (by the co-associativity of H the components on which we apply Δ each time do not matter). The antipode map of H will be denoted by S.

As is well known (see, e.g., [4]), there is a Hopf module isomorphism

$$\bigotimes^{n+1} H \otimes_H k \simeq \bigotimes^{n} H$$

given by

$$h_1 \otimes h_2 \otimes \cdots \otimes h_{n+1} \otimes 1 \mapsto (h_1 \otimes h_2 \otimes \cdots \otimes h_n)\Delta_n\big(S(h_{n+1})\big).$$

5. The Schur multiplier of F/I'

In this section we work over $\mathbb{Z}$. Let L be a finitely generated free Lie algebra over $\mathbb{Z}$, and I an ideal of L such that L/I is commutative and torsion free. We shall compute some properties of the Schur multiplier of L/I'. In particular we shall show that it has a direct summand that is free abelian of infinite rank. This direct summand will be used later. It should be noted that in the case $I = L'$ this was computed by Kuz'min [9], and his results will be used.

If L/I is finitely generated torsion free abelian then we can assume that $L = \langle x_1, \ldots, x_n\rangle$ and $I = \langle L', x_{m+1}, \ldots, x_n\rangle$. If $m = 0$ then $L/I = 0$ so we can assume that $m > 0$. By a slight abuse of notation we identify x_i with its image in L/I. The enveloping algebra of L/I will simply be $\mathbb{Z}[x_1, \ldots, x_m]$, the commutative ring of polynomials in m variables. Since $m > 0$ we know that $U(L/I)$, as an additive group, is free abelian of infinite rank.

We wish to find some of the structure of L/I', and for that we will need the following lemma.

Lemma 5.1. *Let $R = S[x_1, \ldots, x_k]$ be the commutative polynomial ring in k variables over a commutative domain S. Let $d \geqslant 0$, let $1 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant k$ and let $\mu_{i_1,\ldots,i_d} \in S[x_{i_1}, \ldots, x_k] \subseteq R$. If*

$$\sum_{i_1 \leqslant \cdots \leqslant i_d \leqslant k} \mu_{i_1,\ldots,i_d} x_{i_1} \cdots x_{i_d} = 0$$

then $\mu_{i_1,\ldots,i_d} = 0$.

Proof. We prove the lemma with a double induction. First we prove for the case $d = 1$. The base of the first induction is $k = 1$. In that case the equation is $\mu_1 x_1 = 0$, and since R is a domain, obviously $\mu_1 = 0$. Suppose we know that the lemma is true for $d = 1$ and for $k - 1$. The equation is of the form

$$0 = \sum_{i=1}^{m} \mu_i x_i = \mu_1 x_1 + \sum_{i=2}^{k} \mu_i x_i.$$

However, for $i > 1$, $\mu_i x_i \in S[x_2, \dots, x_k]$ and it is easy to see that as an abelian group $R = S[x_2, \dots, x_k] \oplus Rx_1$. From the sum above we can see that $\mu_1 x_1 = 0$ so $\mu_1 = 0$. We thus get an equation with only $k-1$ variables, and the induction hypothesis shows that $\mu_i = 0$ for all i.

We now need to prove the lemma for general d. Once again we can start with the case $k = 1$, and we will get the equation $\mu_{1,\dots,1} x_1^d = 0$, so $\mu_{1,\dots,1} = 0$. Assume we know the lemma for $(d-1, k)$, and for $(d, k-1)$. We have an equation of the form

$$\sum_{i_1,\dots,i_{d-1}} \mu_{1,i_1,\dots,i_{d-1}} x_1 x_{i_1} \cdots x_{d-1} + \sum_{i_1>1} \mu_{i_1,\dots,i_d} x_{i_1} \cdots x_{i_d} = 0.$$

Once again from the fact that $R = S[x_2, \dots, x_k] \oplus Rx_1$ we see that in fact we have two equation,

$$\sum_{i_1>1} \mu_{i_1,\dots,i_d} x_{i_1} \cdots x_{i_d} = 0$$

and

$$\sum_{i_1,\dots,i_{d-1}} \mu_{1,i_1,\dots,i_{d-1}} x_{i_1} \cdots x_{d-1} = 0.$$

The first equation is the case $(d, k-1)$, and the second equation is the case $(d-1, k)$. Thus we are done. $\square$

Note that we only required that S be a commutative domain. Thus, we can allow the coefficients to also involve other variables, since the ring of commutative polynomials is a domain.

We can now find some structure in $H_2(L/I', \mathbb{Z}) = I'/[I', L]$. Recall that I is generated as an ideal by $\{[x_i, x_j]\} \cup \{x_i\colon\ i > m\}$. It can easily be seen that in fact $I = \langle \{x_i\colon\ i > m\} \cup \{[x_i, x_j]\colon\ j < i \leqslant m\} \rangle$.

We choose a linear basis $B = \{l_r\}$ of $U(L/I)$ that consists of all the monomials. Assume that $l_0 = 1$. Let $a_r^i = x_i \cdot l_r$, and $b_r^{ij} = [x_i, x_j] \cdot l_r$. Thus $\{a_0^i, b_0^{ij}\}$ generate I as an ideal.

Since we know the generators of I we can now write a set of linear generators of I'/I^3, whose images in $I'/[I', L]$ will obviously generate it linearly. If $c_1, \dots, c_k, \dots$ generate I then obviously $I'/[I', I]$ is generated by $[c_i \cdot l_1, c_j \cdot l_2]$ where $l_1, l_2 \in B$. We now start finding dependencies between these generators, after dividing by $[I', L]$ (once again we identify these elements with their images in $[I', L]$). Since $[c_i \cdot l_1, c_j \cdot l_2] = -[c_j \cdot l_2, c_i \cdot l_1]$ we can assume that $i \geqslant j$. Also, for any $a, b \in I$ and $x \in \{x_1, \dots, x_n\}$ we have the Jacobi identity

$$\big[a, [b, x]\big] = \big[[a, b], x\big] + \big[[x, a], b\big].$$

But the first summand is in $[I', L]$, so $[a, [b, x]] \equiv -[[a, x], b]$. An easy induction therefore shows that for any $l \in U(L/I)$, $[a \cdot l, b] \equiv [a, b \cdot S(l)]$. Thus, it is enough to consider

$[c_i, c_j \cdot l]$, with $i \geqslant j$ and $l \in B$. We now choose a specific order on the generators of I. We will say that generators of the form a_0^i are greater than generators of the form b_0^{ij}. We will say that if $i < j$ then $a_0^i < a_0^j$. Similarly, if $(i_1, j_1) < (i_2, j_2)$ (taking lexicographical order), then $b_0^{i_1 j_1} < b_0^{i_2 j_2}$. Thus our generators (taking the images in $I'/[I', L]$) are of three possible forms:

$$\alpha_r^{ij} = \left[a_0^i, a_r^j\right], \quad m < j \leqslant i \leqslant n,$$

$$\beta_r^{ijk} = \left[a_0^i, b_r^{jk}\right], \quad 1 \leqslant k < j \leqslant m < i \leqslant n,$$

$$\gamma_r^{ijst} = \left[b_0^{ij}, b_r^{st}\right], \quad j < i \leqslant m,\ t < s \leqslant m,\ (i, j) \geqslant (s, t).$$

One last note is the following. If $k < j < i$, and $l_r = x_k w$ then

$$b_r^{ij} = [x_i, x_j] \cdot x_k w = \big[[x_i, x_j], x_k\big] \cdot w = \big(\big[[x_i, x_k], x_j\big] - \big[[x_j, x_k], x_i\big]\big) \cdot w = b_{r_1}^{ik} - b_{r_2}^{jk},$$

where $l_{r_1} = x_j w$ and $l_{r_2} = x_i w$. Therefore, we can assume that in our generators β_r^{ijk}, $l_r \in \mathbb{Z}[x_k, \ldots, x_n]$.

Lemma 5.2. *The generators α_r^{ij} and β_r^{ijk} with l_r of odd degree have no linear dependencies, and they generate a free abelian group that is a direct summand of $I'/[I', L]$.*

Proof. In order to show that these elements generate a direct summand that is free abelian on them, it is enough to show this under a mapping. In fact, it is enough to show that for each such element there is a mapping under which this is true. Thus, we shall consider the image of the generators under the mapping from the Schur multiplier into $\bigotimes^2(\delta U(L) \otimes_{U(L)} U(L/I)) \otimes_{U(L/I)} k$, and consider this image under all possible projections. Let $e_1, \ldots, e_n$ be a basis of $\delta U(L)$ as a $U(L)$ module. Thus, $\delta U(L) \otimes_{U(L)} U(L/I) = \bigoplus e_i U(L/I)$. Recall that $\phi : I/I' \to \delta U(L) \otimes_{U(L)} U(L/I)$ is the Magnus embedding. It is easily checked that

$$\phi\big(a_r^i\big) = e_i l_r, \quad \phi\big(b_r^{i,j}\big) = (e_i x_j - e_j x_i) l_r.$$

When computing the image of the generators of $I'/[I', L]$ we will first compute their image in $\bigotimes^2(\delta U(L) \otimes_{U(L)} U(L/I))$, and then compute their image after using the Hopf module isomorphism, which we denote π. From here on, in this section, the mapping p_{x_i, x_j} will include the application of π after the projections. It now follows that

$$\alpha_r^{ij} \mapsto (e_i \otimes e_j l_r - e_j l_r \otimes e_i) \otimes 1,$$

$$\beta_r^{ijk} \mapsto \big(e_i \otimes (e_j x_k - e_k x_j) l_r - (e_j x_k - e_k x_j) l_r \otimes e_i\big) \otimes 1,$$

$$\gamma_r^{ijst} \mapsto \big((e_i x_j - e_j x_i) \otimes (e_s x_t - e_t x_s) l_r - (e_s x_t - e_t x_s) l_r \otimes (e_i x_j - e_j x_i)\big) \otimes 1.$$

We can see that if we take $m < j_0 \leqslant i_0 \leqslant n$ then $p_{x_{i_0},x_{j_0}}(\beta_r^{ijk}) = 0$, and $p_{x_{i_0},x_{j_0}}(\gamma_r^{ijst}) = 0$. If $j_0 < i_0$ then $p_{x_{i_0},x_{j_0}}(\alpha_r^{ij}) = \delta_{ii_0}\delta_{jj_0}S(l_r)$. If $i_0 = j_0$ then $p_{x_{i_0},x_{i_0}}(\alpha_r^{ij}) = \delta_{ii_0}\delta_{ji_0}(S(l_r) - l_r)$. Note that for l_r of even degree $S(l_r) - l_r = 0$, but for l_r of odd degree $S(l_r) - l_r = 2S(l_r)$. We claim that there are no dependencies between these elements, and this will also show that the elements α_r^{ij}, where l_r is of odd degree, generate a direct summand of $I'/[I', L]$. Suppose that for all r in the following sum l_r is of odd degree. If

$$\phi\left(\sum_{i,j,r} n_r^{ij}\alpha_r^{ij}\right) = 0,$$

then for all $i_0 > m$

$$0 = p_{i_0,i_0}\left(\sum_{i,j,r} n_r^{ij}\alpha_r^{ij}\right) = \sum_r n_r^{i_0i_0}2S(l_r).$$

However, the $\{l_r\}$ are part of a linear basis of $U(L/I)$ (and hence also the elements $\{S(l_r)\}$) so $n_r^{ii} = 0$ for all r and $i > m$. In addition, for all $m < j_0 < i_0$

$$0 = p_{i_0,j_0}\left(\sum_{i,j,r} n_r^{ij}\alpha_r^{ij}\right) = \sum_r n_r^{i_0j_0}S(l_r),$$

so obviously $n_r^{ij} = 0$ for all n, i, j. Thus we see that the subgroup generated by $\{\alpha_r^{ij}\}$ such that l_r is of odd degree is free abelian with this set as a basis. In fact, for $i \neq j$ we can take any l, not only those of odd degree.

Consider now the case $j_0 \leqslant m < i_0$, then

$$p_{i_0,j_0}(\alpha_r^{ij}) = 0,$$
$$p_{i_0,j_0}(\gamma_r^{ijst}) = 0,$$
$$p_{i_0,j_0}(\beta_r^{ijk}) = \delta_{ii_0}\left(\delta_{kj_0}x_jS(l_r) - \delta_{jj_0}x_kS(l_r)\right).$$

We claim that there are no dependencies between the elements β_r^{ijk} with l_r of odd degree, and this will show that these elements generate a direct summand. In fact, we can take l_r to be of any degree. Indeed, if

$$\phi\left(\sum_{i,j,k,r} n_r^{ijk}\beta_r^{ijk}\right) = 0$$

then for all $j_0 \leqslant m < i_0$ we have

$$0 = p_{i_0,j_0}\left(\sum_{i,j,k,r} n_r^{ijk}\beta_r^{ijk}\right) = \sum_{r,k<j_0} n_r^{i_0kj_0}x_{j_0}S(l_r) - \sum_{r,k>j_0} n_r^{i_0j_0k}x_{j_0}S(l_r).$$

Recall that for specific $j > k$, we have $l_r \in \mathbb{Z}[x_k, \dots, x_m]$. We claim that $n_r^{ijk} = 0$ for all r, i, j, k. We shall prove this for each i separately. We prove by induction on j. If we take $i_0 = i$, $j_0 = 1$ we get the equation $0 = \sum_{r,k>1} n_r^{i1k} x_k S(l_r)$. Let us denote $\mu_k = \sum_r n_r^{i1k} S(l_r)$. We know that $\mu_k \in \mathbb{Z}[x_k, \dots, x_m]$, and we get the equation

$$\sum_k \mu_k x_k = 0.$$

From Lemma 5.1 we see that $\mu_k = 0$, so $n_r^{i1k} = 0$ for all r, i, k. Suppose we have shown that $n_r^{ij'k} = 0$ for all r, i, k and $j' < j$. If we take $i_0 = i$ and $j_0 = j$ we get the equation

$$0 = \sum_{r,k>j} n_r^{ijk} x_k S(l_r) - \sum_{r,k<j} n_r^{ikj} x_k S(l_r)$$

but for $k < j$, $n_r^{ikj} = 0$. Let $\mu_k = \sum_{r,k>j} n_r^{ijk} S(l_r)$. Once again $\mu_k \in \mathbb{Z}[x_k, \dots, x_m]$, and we have the equation

$$\sum_{k>j} \mu_k x_k = 0,$$

so another application of Lemma 5.1 finishes the proof. $\square$

We are left with the elements of the form γ_r^{ijst}. Consider the subalgebra $L_1 = \langle x_1, \dots, x_m \rangle \subseteq L$. The elements γ_r^{ijst} are all in L_1''. Further, let J be the ideal generated by $x_{m+1}, \dots, x_n$, then because $L = L_1 \oplus J$, we have $[I', L] \cap L_1 = [L_1'', L_1]$, so $L_1''/[L_1'', L_1]$ is naturally embedded as a direct summand in $L''/[L'', L]$. However, this is simply the free center-by-metabelian Lie algebra on m variables, which was completely described by Kuz'min. Thus we have

Lemma 5.3 (Kuz'min [9]). *The elements γ_r^{ijst} with $i > j$, $s > t$, $i \geqslant s$, $j \geqslant t$, $l_r \in \mathbb{Z}[x_j, \dots, x_m]$ and l_r of odd degree are linearly independent in $I'/[I', L]$ and form a free abelian direct summand. If $d > m$ is odd, then these elements, with* $\deg(l_r) = d$*, generate all the elements of weight $d + 4$.*

6. Computations

We can now prove Theorem 1.1, i.e. show that $H_2(L/[I, J], k)$ is not finitely generated by exhibiting an infinite number of elements of the Schur multiplier, whose images in $\bigotimes^n U(L/K)$ are linearly independent. We shall deal with several cases. In each of them we shall construct elements of $H_2(L/[I, J], k)$ that have one parameter l, where $l \in U(L/K)$. In other words we shall construct a k-linear map $f : U(L/K) \to H_2(L/[I, J], k) \to \bigotimes^n U(L/K)$. It is obviously enough to show that $\operatorname{Im} f$ is not finite dimensional, for instance by proving that it has elements of unbounded degree.

Recall the elements $\alpha \in I$, $x \in X$, $\beta \in J$ and $x_1, \dots, x_n \in X$ such that $a = p_x(\alpha)$ and $b = (p_{x_1} \otimes \cdots \otimes p_{x_n})(\beta)$ were non-zero, and consider all elements of the form $[\alpha \cdot l, \beta]$, where l is any element of $\delta U(L/I)$. Obviously this element is in $[I, J]$. Consider its image, using the mapping $\phi_{x,x_1,\dots,x_n}$. Since $b \in \bigotimes^n U(L/K)$ we will use the notation $b = \sum_i b_i^1 \otimes b_i^2$, where $b_i^1 \in \bigotimes^{n-1} U(L/K)$, $b_i^2 \in U(L/K)$, and the b_i^2 are linearly independent. Thus, under the Hopf module isomorphism

$$f(l) = \sum_i \bigl(al \otimes b_i^1\bigr)\Delta_n S\bigl(b_i^2\bigr) - b'\Delta_n\bigl(S(l)\bigr)\Delta_n\bigl(S(a')\bigr).$$

(Note that we use a', b' because the projections on the various coordinates might be different.)

We now consider several cases.

Case I. Suppose $n > 1$. We will consider the last coordinate in the tensor product. Since $K \neq L$ we can assume that there is $y \in X$ such that $y \notin K$. We claim that for $l_m = y^m$ of high degree, the elements $\{f(l_m)\}$ are linearly independent. If Z is a homogeneous linear basis of $U(L/K)$ then $\{(z_1 \otimes \cdots \otimes z_n)\colon z_i \in Z\}$ is a linear basis of $\bigotimes^n U(L/K)$. Let $w \in \bigotimes^n U(L/K)$, so we can write

$$w = \sum \alpha_i \bigl(z_i^1 \otimes \cdots \otimes z_i^n\bigr)$$

with $\alpha_i \neq 0$. We can define the maximal degree of the last coordinate to be $\max\{\deg(z_i^n)\}$, and denote it $md(w)$. It is easy to see that md does not depend on the choice of Z. In fact, if we write a as any sum of linearly independent tensor products (with non-zero coefficients), then $md(a)$ will be the highest degree appearing in the last coordinate of the tensor products. Obviously if we show that for large m, $md(f(l_m)) = m + g$, where g is constant, then we are done. The last coordinates of $\sum_i (al_m \otimes b_i^1)\Delta_n S(b_i^2)$ will be constant, as they do not depend on l_m. It is easy to see that the leading term of the last coordinate of $\Delta_n(y)$ is $\bar{y}$. Since $b' \in \bigotimes^n U(L/K)$ we will use the notation $b' = \sum_i c_i^1 \otimes c_i^2$, where $c_i^1 \in \bigotimes^{n-1} U(L/K)$, $c_i^2 \in U(L/K)$, and the c_i^2 are linearly independent. The leading terms of the last coordinate of $b'\Delta_n(S(l_m))\Delta_n(S(a'))$ will therefore be of the form $\overline{c_i^2 S(l_m) S(a')}$, and hence linearly independent, so that $md(b'\Delta_n(S(l_m))\Delta_n(S(a'))) = \deg(c_i^2) + m + \deg(a')$. For large m, the maximal degree appearing in the last coordinates of the constant portion will be majorized by these terms. Thus, if $g = \max\{\deg(c_i^2) + \deg(a')\}$ then $md(f(l_m)) = m + g$.

Case II. Suppose $n = 1$ and $\operatorname{char}(k) \neq 2$. Since $n = 1$ we can write $f(l) = alS(b) - b'S(l)S(a')$. Let $y \in X$ such that $y \notin K$. Let $d = \max\{\deg(aS(b)), \deg(b'S(a'))\}$. It is easy to show that for any $c \in U(L/I)$, $cy^i \equiv y^i c$ modulo terms of lower degree. If we take $l = y^i$, clearly $f(l) = ay^i S(b) - (-1)^i b' y^i S(a')$, and $\deg(f(l)) \leqslant d + i$. Let g_i be the image of $f(y^i)$ in U_{d+i}/U_{d+i-1}, and let h_i be the image of $aS(b) - (-1)^i b'S(a')$ in U_d/U_{d-1}, then $g_i = h_i y^i$. It is therefore enough to show that h_i is non-zero for unbounded values of i. If $\overline{aS(b)} \neq \pm\overline{b'S(a')}$ then $h_i \neq 0$ for all i. If $\overline{aS(b)} = \epsilon\overline{b'S(a')}$, $\epsilon = \pm 1$, then for i of the correct parity we will get $h_i = 2\overline{aS(b)}$, and since $\mathrm{char}(k) \neq 2$, this means that $h_i \neq 0$, and this will be true for unbounded values of i.

Case III. The only case left is $\mathrm{char}(k) = 2$ and $n = 1$. We will assume that L/K is not commutative, and that the image of f is finite dimensional, and reach a contradiction. The case that L/K is commutative will be dealt with in Section 7. Thus, if $y \in U(L/I)$, $\{f(y^i)\}$ will have bounded degree, say m. Denote $v_i = ay^i S(b) - b' y^i S(a')$, so $\deg(v_i) < m$. Let $d = \deg(aS(b))$. Clearly $\deg(b'S(a')) = d$, otherwise v_i would not have bounded degree. Since L/K is not commutative, there exist $x, y \in X$ such that $[x, y] \notin K$, i.e. $[x, y] \neq 0$ in $U(L/K)$ (and clearly $y \neq 0$ in $U(L/K)$). Note that for any $a, b \in U(L/K)$, $\deg(ab - ba) < \deg(a) + \deg(b)$. An easy induction also shows that $xy^i - y^i x = iy^{i-1}[x, y]$. Consider $l_i = xy^i$, and take $i > m + 1$. Obviously $S(l_i) = y^i x$, so

$$\begin{aligned} f(l_i) &= axy^i S(b) - b' y^i x S(a') \\ &= (ax - xa) y^i S(b) + x\big(ay^i S(b) - b' y^i S(a')\big) \\ &\quad + (xb' - b'x) y^i S(a') + b'\big(xy^i - y^i x\big) S(a') \\ &= (ax - xa) y^i S(b) + x v_i + (xb' - b'x) y^i S(a') + b'\big(xy^i - y^i x\big) S(a'). \end{aligned}$$

Thus $\deg(f(l_i)) \leqslant d + i$ (and $\deg(xv_i) < d + i$). Let g_i be the image of $f(l_i)$ in U_{d+i}/U_{d+i-1}, and let h_i be the image of

$$y(ax - xa)S(b) + y(xb' - b'x)S(a') + ib'S(a')[y, x]$$

in U_{d+1}/U_d. However, U_{d+i}/U_{d+i-1} is commutative, so we see that $g_i = h_i y^{i-1}$, and once again we need only show that $h_i \neq 0$ for unbounded i. But clearly, since $\deg(b'S(a')[y, x]) = d + 1$, there must be at least one parity of i for which this is true.

7. Characteristic 2

In this section we shall deal with the case $\mathrm{char}(k) = 2$. First we finish the proof of Theorem 1.1 by considering the case $\mathrm{char}(k) = 2$ and L/K commutative. Afterwards we shall correct a slight error made in [4].

Suppose L is generated by n variables $x_1, \ldots, x_n$, and $I + J = K \neq L$ such that L/K is commutative. Since $L' \subseteq K$ we can assume that $K = \langle L', x_{m+1}, \ldots, x_n \rangle$, where $0 < m \leqslant n$. We first make use of the universal coefficient theorem. Let L_1 be the free Lie algebra over $\mathbb{Z}$ with basis $y_1, \ldots, y_n$ and let $K_1 = \langle L_1', y_{m+1}, \ldots, y_n \rangle$. Thus

$$\begin{aligned} 0 &\to H_2\big(L_1/K_1', \mathbb{Z}\big) \otimes_{\mathbb{Z}} k \to H_2\big(L_1/K_1' \otimes_{\mathbb{Z}} k, k\big) \\ &\to \mathrm{Tor}_1^{\mathbb{Z}}\big(H_1\big(L_1/K_1', \mathbb{Z}\big), k\big) \to 0 \end{aligned}$$

is exact. Since $H_1(L_1/K_1', \mathbb{Z}) = (L_1/K_1')_{ab}$ is a free $\mathbb{Z}$ module then $\mathrm{Tor}_1^{\mathbb{Z}}(H_1(L_1/K_1', \mathbb{Z}), k) = 0$.

Obviously, $L/K' \cong L_1/K_1 \otimes_{\mathbb{Z}} k$, so we have

$$K'/[K', L] \cong K_1'/\big[K_1', L_1\big] \otimes_{\mathbb{Z}} k.$$

Thus, all the results of Section 5 apply also to $K'/[K', L]$. We will identify x_i with its image in L/K. Since $U(L/K)$ is the commutative ring of polynomials in $x_1, \dots, x_m$, we can take a linear basis of $U(L/K)$ to be the monomials. Thus, if $\{l_r\}$ is such a basis, we denote (considering the images in $K'/[K', L]$)

$$\alpha_r^{ij} = [x_i, x_j \cdot l_r], \quad m < j \leqslant i \leqslant n,$$
$$\beta_r^{ijk} = \big[x_i, [x_j, x_k]\big] \cdot l_r, \quad k < j \leqslant m < i \leqslant n,\ l_r \in \mathbb{Z}[x_k, \dots, x_m],$$
$$\gamma_r^{ijst} = \big[[x_i, x_j], [x_s, x_t]\big] \cdot l_r, l_r \in k[x_{\min(j,t)}, \dots, x_m]$$

and all of these elements are non-zero and linearly independent in $K'/[K', L]$.

In addition, we have a linear basis of K/K' comprised of the following elements:

$$a_r^i = x_i \cdot l_r, \quad i > m,$$
$$b_r^{ij} = [x_i, x_j \cdot l_r], \quad j \leqslant i \leqslant m,\ l_r \in k[x_j, \dots, x_m].$$

We discern two cases. In the first case, $m < n$, i.e. $K \neq L'$, and in the second $m = n$, i.e. $K = L'$. If $K \neq L'$, let $M = K/K'$ and $N = K'/[K', L]$. We define the following subspaces of N:

$$W_d = \big\langle \big\{\alpha_r^{ij}, \beta_r^{ijk}, \gamma_r^{ijst}\colon \deg(l_r) \geqslant d\big\}\big\rangle,$$
$$V_d = W_{d+1} + \big\langle \big\{\alpha_r^{ij}\colon i \geqslant j > m+1,\ \deg(l_r) = d\big\}\big\rangle$$
$$+ \big\langle \big\{\beta_r^{ijk}\colon i > m+1,\ \deg(l_r) = d\big\}\big\rangle + \big\langle \big\{\gamma_r^{ijst}\colon \deg(l_r) = d\big\}\big\rangle,$$
$$U_d = V_d + \big\langle \big\{\alpha_r^{i(m+1)}\colon i > m+1,\ \deg(l_r) = d\big\}\big\rangle + \big\langle \big\{\beta_r^{(m+1)jk}\colon \deg(l_r) = d\big\}\big\rangle$$

and the following subspaces of M:

$$A_d = \big\langle \big\{a_r^i, b_r^{ij}\colon \deg(l_r) \geqslant d\big\}\big\rangle = \delta U(L/K)^d M,$$
$$B_d = A_{d+1} + \big\langle \big\{a_r^i\colon i > m+1,\ \deg(l_r) = d\big\}\big\rangle + \big\langle \big\{b_r^{ij}\colon \deg(l_r) = d\big\}\big\rangle.$$

Note that $\bigcap A_d = 0$.

Since $M = K/K'$, then by an abuse of notation we can say that there is a mapping $[M, M] \to N$. Thus, if $x, y \in M$ we can refer to $[x, y] \in N$, and this image indeed does not depend on the representatives we choose. We can thus also discuss the commutator subgroup generated by two subgroups of M. In the following lemma we use this notation:

Lemma 7.1. *For all d, $[M, A_d] \subseteq W_d$, $[A_d, B_0] \subseteq U_d$ and $[B_d, B_0] \subseteq V_d$. In addition, if d is odd and $y \in B_d \setminus A_{d+1}$ then $[y, x_{m+1}] \notin V_d$. If d is odd and $y \in A_d \setminus B_d$ then $[x_{m+1}, y] \notin U_d$.*

Proof. It is an easy induction to show that $[A_d, A_g] \subseteq W_{d+g}$, and since $A_0 = M$, the first statement is obvious. To show that $[A_d, B_0] \subseteq U_d$, let $x \in A_d$, $y \in B_0$. Obviously

$$y = \sum_{i=m+2}^{n} e_i a_0^i + \sum_{j,k} f_{ij} b_0^{jk} + w,$$

where $w \in A_1$. Let $y_1 = y - w$. In addition,

$$x = \sum_{i,r:\ \deg(l_r)=d} h_r^i a_r^i + \sum_{j,k,r:\ \deg(l_r)=d} p_r^{jk} b_r^{jk} + v,$$

where $v \in A_{d+1}$. Let $x_1 = x - v$. Clearly $[x, w] \in W_{d+1}$, so $[x, w] \in U_d$. In a similar fashion it can be seen that $[x, y] - [x_1, y_1] \in U_d$. But the only part of $[x_1, y_1]$ that might not be in U_d is a sum of elements of the form α_r^{ij} with $i > m + 1$ (it might be that $j > i$, but that only adds a sign), and since they do belong to U_d, we are done. A very similar argument shows that $[B_d, B_0] \subseteq V_d$. Suppose that $y \in B_d \setminus A_{d+1}$, i.e.

$$y = \sum_{i>m+1,r:\ \deg(l_r)=d} c_r^i a_r^i + \sum_{j,k,r:\ \deg(l_r)=d} h_r^{jk} b_r^{jk} + v,$$

where $v \in A_{d+1}$, and at least one of the c_r^i, h_r^{jk} is non-zero. Obviously $[x_{m+1}, v] \in V_d$, so we can look at

$$[x_{m+1}, y - v] = -\sum_{i>m+1,r:\ \deg(l_r)=d} c_r^i \alpha_r^{i(m+1)} - \sum_{j,k,r:\ \deg(l_r)=d} h_r^{jk} \beta_r^{(m+1)jk}.$$

From the linear independence of all elements with l_r of odd degree, we can see that U_d/V_d (as a vector space) has a basis comprised of the elements $\{\alpha_r^{i(m+1)}\colon i > m+1, \deg(l_r) = d\} \cup \{\beta_r^{(m+1)jk}\colon \deg(l_r) = d\}$. Thus, one can see that $[x_{m+1}, y - v] \notin V_d$. A similar argument shows the rest of the lemma. $\square$

Since $x_{m+1} \in K$ then $x_{m+1} = a + b$ where $a \in I$ and $b \in J$. Thus we can assume without loss of generality that $a = \epsilon x_{m+1} + w_1$ where $w_1 \in B_0$ and $0 \neq \epsilon \in k$. Thus $x_{m+1} + w \in I$ where $w = \epsilon^{-1} w_1 \in B_0$. In addition, since $J \nsubseteq K'$, there is $0 \neq b \in J$. Thus, there is some d such that $b \in A_d \setminus A_{d+1}$. Let $b_g = b \cdot x_1^g$ for $g \geqslant 0$. Clearly $b_g \in J$, and $b_g \in A_{d+g} \setminus A_{d+g+1}$. Suppose that $b \in B_d$, and hence $b_g \in B_{d+g}$. In this case, $[b_g, a] = [b_g, x_{m+1}] + [b_g, w]$. Since $w \in B_0$, we have $[b_g, w] \in V_{d+g}$, but for $g + d$ odd, $[b_g, x_{m+1}] \notin V_{d+g}$. Thus, $[b_g, a] \neq 0$, and since the elements of odd degree are linearly independent, we have displayed an infinite number of non-zero elements that are linearly independent, so the Schur multiplier is infinite dimensional. If $b \notin B_d$ then $b_g \notin B_{g+d}$. However, $[b_g, w] \in$

$[A_{d+g}, B_0] \subseteq U_{d+g}$, but for $d+g$ odd $[b_g, x_{m+1}] \notin U_{d+g}$, so once again we have shown that the Schur multiplier is infinite dimensional.

We are left with the case $m=n$, i.e. $K=L'$. We define the following subspaces of N:

$$W_d = \langle \{ \gamma_r^{ijst}\colon \deg(l_r) \geqslant d \} \rangle,$$
$$V_d = W_{d+1} + \langle \{ \gamma_r^{ijst}\colon (i,j) \geqslant (s,t) > (2,1),\ \deg(l_r) = d \} \rangle,$$
$$U_d = V_d + \langle \{ \gamma_r^{ij21}\colon (i,j) > (2,1),\ \deg(l_r) = d \} \rangle$$

and the following subspaces of M:

$$A_d = \langle \{ b_r^{ij}\colon \deg(l_r) \geqslant d \} \rangle = \delta U(L/K)^d M,$$
$$B_d = A_{d+1} + \langle \{ b_r^{ij}\colon (i,j) > (2,1),\ \deg(l_r) = d \} \rangle.$$

Once again, with the same abuse of notation, it is easy to check that $[M, A_d] \subseteq W_d$, $[A_d, B_0] \subseteq U_d$ and $[B_d, B_0] \subseteq V_d$. In addition, Let $z = [x_2, x_1]$ and $d > n$. If d is odd and $y \in B_d \setminus A_{d+1}$ then $[z, y] \notin V_d$. If d is odd and $y \in A_d \setminus B_d$ then $[z, y] \notin U_d$. Once again, $\bigcap A_d = 0$. It should be noted that here we use the fact that we chose $z = [x_2, x_1]$, for then $[z, b_r^{ij}] = \pm\gamma_r^{ij21}$ is always one of the elements that is known to be linearly independent, since perforce $i \geqslant 2$, $j \geqslant 1$ and the restriction $l_r \in k[x_j, \ldots, x_n]$ is sufficient.

Since $z \in K$ then $z = a + b$ where $a \in I$ and $b \in J$. We can assume without loss of generality that $a = \epsilon z + w_1$ where $w_1 \in B_0$ and $0 \neq \epsilon \in k$. Thus $z + w \in I$ where $w = \epsilon^{-1} w_1 \in B_0$. In addition, since $J \nsubseteq K'$, there is $0 \neq b \in J$. Thus, there is some d such that $b \in A_d \setminus A_{d+1}$. Let $b_g = b \cdot x_1^g$ for $g \geqslant 0$. Clearly $b_g \in J$, and $b_g \in A_{d+g} \setminus A_{d+g+1}$. Suppose that $b \in B_d$, and hence $b_g \in B_{d+g}$. In this case, $[b_g, a] = [b_g, z] + [b_g, w]$. Since $w \in B_0$, $[b_g, w] \in [B_{d+g}, B_0] \subseteq V_{d+g}$ but for $g > n$ and $g + d$ odd, $[b_g, z] \notin V_{d+g}$. For the same reasons as before, we have shown that the Schur multiplier is infinite dimensional. If $b \notin B_d$ then $b_g \notin B_{g+d}$. However, $[b_g, w] \in [A_{d+g}, B_0] \subseteq U_{d+g}$, but for $g > n$ and $g + d$ odd, $[b_g, z] \notin U_{d+g}$, so once again the Schur multiplier is infinite dimensional, and this finishes the proof of Theorem 1.1.

In [4] a similar problem arose with characteristic 2, where L/I is commutative, and we consider the Schur multiplier of L/I'. As in our case, L/I can be described as $L_1/I_1 \otimes k$, where L_1 is a free Lie algebra over $\mathbb{Z}$. The universal coefficient theorem was also employed. However, the argument used was that since the Schur multiplier of $L_1/I_1' \otimes \mathbb{Q}$ is infinite dimensional, then the Schur multiplier of L_1/I_1' has infinite rank. While this is true, it is not enough in order to show that after tensoring with k we will get an infinite-dimensional vector space. Indeed, take a direct sum of an infinite number of copies of $\mathbb{Z}[1/2]$. When tensored with $\mathbb{Q}$ this will be infinite dimensional. However, when tensored with $\mathbb{Z}/2\mathbb{Z}$ the result will be zero.

The way around this problem is that it was shown in Section 5 that in fact the Schur multiplier of L_1/I_1' has a direct summand that is free abelian of infinite rank. Now, of course, when tensoring with any field, we get an infinite-dimensional vector space.

8. An example

First, we give a short proof for the following lemma, which was part of Theorem 1.1.

Lemma 8.1. *Let L be a finitely generated free Lie algebra over a field, and let I, J be ideals of L such that I, J are finitely generated as ideals, and $I + J = L$. Then $[I, J]$ is finitely generated as an ideal.*

Proof. Choose a generating set $X = x_1, \dots, x_k$ for L. Since L is finitely generated, $L = I + J$, and I, J can be finitely generated as ideals, there are elements $a_1, \dots, a_n \in I$ and $b_1, \dots, b_m \in J$ such that I is generated as an ideal by $\{a_i\}$, J is generated as an ideal by $\{b_j\}$ and for each x_i we can write $x_i = a_i + b_i$. We now claim that $[I, J]$ is generated as an ideal by $\{[a_i, b_j]\}$. Indeed, let $K = \langle [a_i, b_j] \rangle$ be the ideal generated by these elements. Obviously we need to show that $K = [I, J]$. An element of I is the sum of elements of the form $a_i \cdot l_1$ where l_1 is a monomial in $U(L)$, and an element of J is a sum of elements of the form $b_j \cdot l_2$ where l_2 is also a monomial in $U(L)$. Thus, it is enough to show that $[a_i \cdot l_1, b_j \cdot l_2] \in K$ for all i, j, l_1, l_2. We prove this by an induction on $\deg(l_1) + \deg(l_2)$. The claim is obvious for $\deg(l_1) + \deg(l_2) = 0$. Assume $\deg(l_1) > 0$. Thus, $l_1 = w x_{i_0}$ where $\deg(w) = \deg(l_1) - 1$ and $x_{i_0} \in X$, so that $x_{i_0} = a_{i_0} + b_{i_0}$. Note that $a_i \cdot (w x_{i_0}) = [a_i \cdot w, x_{i_0}]$, so

$$[a_i \cdot l_1, b_j \cdot l_2] = \big[[a_i \cdot w, a_{i_0}], b_j \cdot l_2\big] + \big[[a_i \cdot w, b_{i_0}], b_j \cdot l_2\big].$$

By the induction hypothesis, the second summand is in K. By Using the Jacobi identity we see that

$$\big[[a_i \cdot w, a_{i_0}], b_j \cdot l_2\big] = \big[[b_j \cdot l_2, a_{i_0}], a_i \cdot w\big] + \big[[a_i \cdot w, b_j \cdot l_2], a_{i_0}\big],$$

and it is easy to see that by the induction hypothesis, both summands are in K. If $\deg(l_1) = 0$ a similar argument reduces the degree of l_2. $\square$

An interesting question is, what happens when $I + J = L$ but they are not both finitely generated. For instance, if we take $I = L$, and J not finitely generated as an ideal, can $[J, L]$ be finitely generated? In fact, if we take J such that $H_2(L/J, k)$ is infinite dimensional, yet $[J, L]$ is finitely generated, then the exact series

$$0 \to J/[J, L] \to L/[J, L] \to L/J \to 0$$

gives us a finitely presented Lie algebra with an infinite-dimensional center. The following lemma shows that the converse is also true. However, it is known that if $\mathrm{char}(k) \neq 2$ there are such finitely presented Lie algebras over k with an infinite-dimensional center [1,5]. Thus, if $\mathrm{char}(k) \neq 2$ there exists an ideal J such that the Schur multiplier of L/J is not finitely presented, yet $[J, L]$ is finitely generated as an ideal.

Lemma 8.2. *Let L be a finitely generated free Lie algebra over a field k. There exists an ideal $J \subseteq L$ such that J is not finitely generated as an ideal, and in fact the Schur multiplier* *of L/J is infinite dimensional, yet $[J, L]$ is finitely generated as an ideal if and only if there exists a finitely presented Lie algebra over k with an infinite-dimensional center.*

Proof. By the argument above it is clear that if we have such an ideal J, then $L/[J, L]$ is the required algebra. Suppose that we have a finitely presented Lie algebra over k, $\mathcal{G}$, such that $Z(\mathcal{G})$ is infinite dimensional. Suppose that we have a presentation

$$0 \to I \to L \to \mathcal{G} \to 0.$$

Since $\mathcal{G}$ is finitely presented we know that L is finitely generated and I is finitely generated as an ideal. Thus, by the previous lemma, $[I, L]$ is finitely generated as an ideal. Let J be the preimage of the center of $\mathcal{G}$ in L. We claim that J is the required ideal. Since J is the preimage of the center we know that $[J, L] \subseteq I$. Thus, $[I, L] \subseteq [J, L] \subseteq I$. Since both I and $[I, L]$ are finitely generated, and $I/[I, L]$ is finite dimensional, then $[J, L]$ is finitely generated. We need to show that $J/[J, L]$ is not finite dimensional. We know that J/I is infinite dimensional. Consider the exact sequence

$$0 \to I/[J, L] \to J/[J, L] \to J/I \to 0.$$

Thus, it is enough to show that $I/[J, L]$ is finite dimensional. However, $I/[J, L]$ is an image of $I/[I, L]$. Since I is finitely generated as an ideal, this is finite dimensional, and we are done. $\square$

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